# Harmonious Labelings of Infinite Graphs

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**Abstract**

In graph theory, graphs are mathematical structures containing a set of vertices and a set of edges that represent the pairwise relationship of objects. A graph can be labeled with a function that assigns a number to each vertex. A harmonious labeling of a graph $G$ assigns positive numbers to the vertices such that the sum of each adjacent vertex label is distinct modulo the number of edges in $G$. In our research we expanded the definition of harmonious labelings to apply to infinite graphs, and investigated which infinite graphs are harmonious by our definition. We also defined and investigated a new type of labeling for both finite and infinite graphs, locally harmonious labeling.

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# 1 Introduction

The first paper in the history of graph theory was the "Seven Bridges of Königsberg." Königsberg, modern day Kaliningrad, was a town in Prussia; its citizens attempted to find a way to cross each of the seven bridges across the Pregel River (see Fig. 1) once and only once.

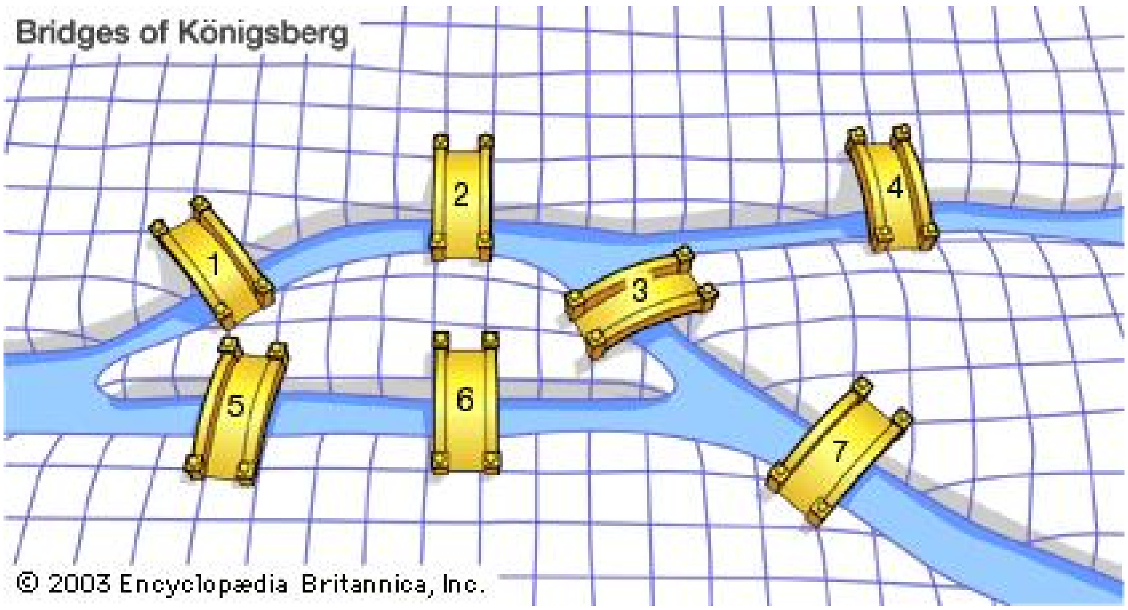


Figure 1: The Seven Bridges of Königsberg [1]

Leonhard Euler showed that it was impossible to do so, leading to the creation of the field of graph theory. Graphs are mathematical structures that capture the pairwise relationship of objects. See Definition 1.1 for a formal description. Because graphs can represent pairwise relations, graph theory has applications in diverse fields including computer science [4], biochemistry [9], and electrical engineering [3]. Graphs are also used to model network flow, for example, and connections between people on social media can be represented as graphs.

**Definition 1.1.** A graph $G = (V, E)$ consists of a nonempty set of vertices, V, and a set of edges, E, which are two element subsets of V (see Fig. 2).

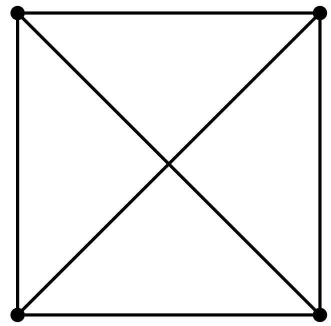

Figure 2: Example of a graph (specifically the complete graph on four vertices, $K_4$).

In graph theory there are many inherent properties that families of graphs have. One problem in graph theory is that of planar embedding. A planar graph is a graph the edges of which do not cross, and the problem of planar embedding seeks to determine which graphs are planar. One of the most famous and important theorems in graph theory is the Four Color Theorem. This states that no planar graph requires more than four colors to color its vertices so that no adjacent vertices have the same color. Finally, there is another interesting property of graphs known as Euler's formula. This states that $v - e + f = 2$ for all planar graphs, where $v$ is the number of vertices, $e$ is the number of edges, and $f$ is the number of faces. Euler's formula has proved its usefulness in a variety of mathematical fields, such as topology.

One thing in graph theory that is of interest is the study of graph labelings. A graph is labeled with a function that takes each vertex and assigns it a number. The first type of labeling that was studied in depth was a *graceful labeling* by Rosa [10]. A graph labeling is graceful when the difference between each adjacent vertex is unique for all edges modulo the number of edges. Many graphs have been shown to be graceful, including all simple graphs with four or less vertices, all paths, and all caterpillars. A famous unsolved problem relating to graceful labelings is the *Graceful*

*Tree Conjecture* (also called the *Ringel-Kotzig Conjecture*), which says that all finite trees are graceful. This result has been proven for infinite graphs [2]. Since Rosa, over 2,000 papers on labelings have been written (see [5] for an extensive survey). An example of one of these papers is [7], which showed that certain families of finite graphs, such as caterpillars, are *harmonious* (see Definition 2.14). Our research focused on extending the definition of harmonious labelings to infinite graphs and then determining which families of infinite graphs are harmonious.

In Section 2 we will define basic concepts relevant to graph theory and graph labelings. In Section 3 we will consider harmonious labeling and the results thereof. In Section 4 we will define locally harmonious labeling and consider which graphs are locally harmonious. Finally, in Section 5 we will conclude our findings.

## 2 Preliminaries

In this section are important definitions and examples relating to the topic of the paper. All of these definitions were obtained from [8].

**Definition 2.1.** A function for which all outputs can only be obtained with a single input is an *injective function*, or an *injection*.

**Definition 2.2.** A function which is injective and for which also all elements in the range can be obtained as outputs is a *bijective function*, or a *bijection*.

**Definition 2.3.** A ∪ B is the *union* of A and B, the set containing all elements which are elements of A or B or both.

**Definition 2.4.** A ⊂ B asserts that A is a *proper subset* of B: every element of A is also an element of B, but A ≠ B. A ⊃ B asserts that A is a *proper superset* of B: every element of B is also an element of A, but A ≠ B.

**Definition 2.5.** Two vertices are *adjacent* if they are connected by an edge. Two edges are *adjacent* if they share a vertex.

**Definition 2.6.** A graph is *bipartite* if it is possible to divide its vertices into two disjoint sets such that there are no edges between any two vertices in the same set.

**Definition 2.7.** A *complete bipartite graph* is a bipartite graph such that every pair of graph vertices in the two sets are adjacent, and is denoted $K_{m,n}$, where m and n are the sizes of the two disjoint sets described in Definition 2.6.

**Definition 2.8.** A graph is *connected* if there is a path from any vertex to any other vertex (see Fig. 3).

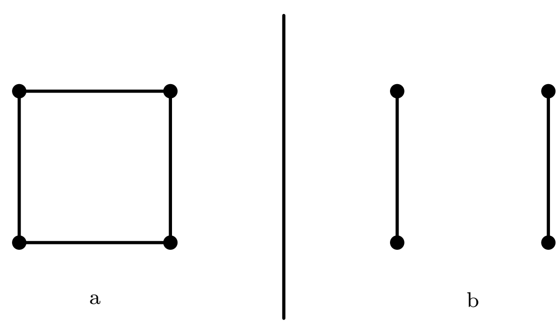


Figure 3: A connected graph (a) and a disconnected graph (b).

**Definition 2.9.** The *degree* of a vertex is the number of edges incident to it, denoted by $d(v_i)$ (see Fig. 4).

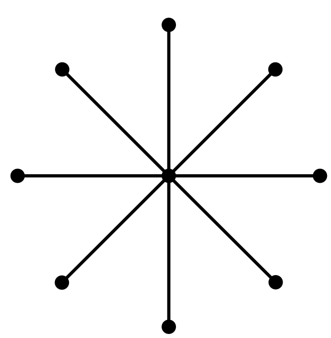

Figure 4: A graph (specifically a star) with the central vertex having degree 8.

**Definition 2.10.** A *path* is a connected graph with all vertices having a maximum degree of two. A path with $n$ vertices is denoted $P_n$ (see Fig. 5).

**Definition 2.11.** A *cycle* is a path that starts and stops at the same vertex, but contains no other repeated vertices. A cycle with $n$ vertices is denoted $C_n$ (see Fig. 6).

**Definition 2.12.** A *tree* is a graph with no cycles (see Fig. 7).

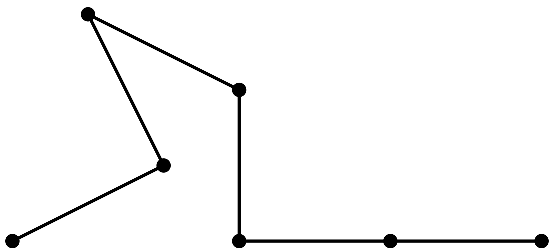

Figure 5: An example of a path.

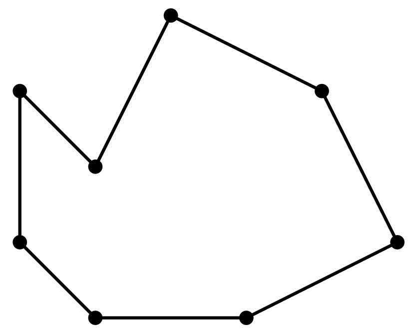

Figure 6: An example of a cycle.

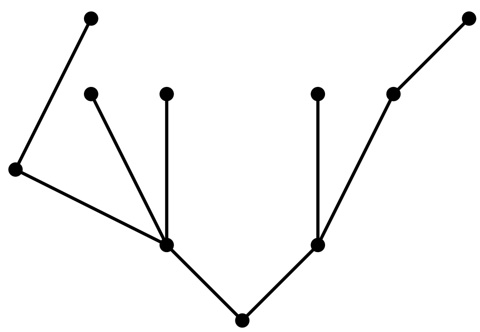

Figure 7: An example of a tree.

In general, a graph is labeled with a function that takes each vertex and assigns it a number. Graph labelings are of great interest to mathematicians and over 2000 papers have been written relating to graph labelings in general or specific graph labelings [5].

**Definition 2.13.** A *graph labeling* of a graph G is a map $\varphi$ from the vertex set of G to a countable set (often nonnegative integers). The label of each edge $(v_i, v_j) \in E$ can then be induced from the labels $\varphi(v_i)$ and $\varphi(v_j)$ of vertices $v_i$ and $v_j$ (see Fig. 8).

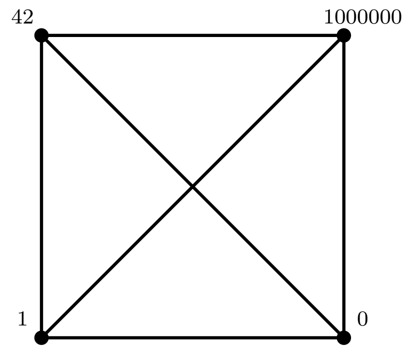


Figure 8: Example of a labeled graph

**Definition 2.14.** A graph $G = (V, E)$ with $e$ edges has a *harmonious labeling* if there exists an injective (see Definition 2.1) function $\varphi$ from the vertices of $G$ to the group of positive integers $\mathbb{Z}^+$ modulo $e$ such that when each edge $(v_i,\, v_j) \in E$ is assigned the label $\varphi(v_i) + \varphi(v_j) \pmod{e}$, the resulting edge labels are distinct. A graph G that meets the above criterion is said to be *harmonious* (see Fig. 9).

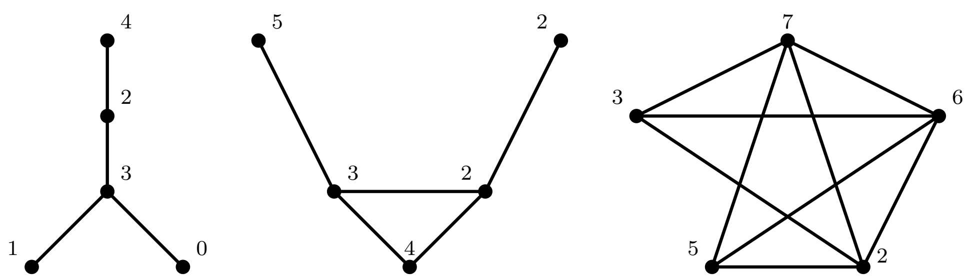


Figure 9: Three labeled harmonious graphs.

Certain finite graphs have been shown to be harmonious, such as caterpillars and odd cycles [7]. Other variations of harmonious labelings, such as *even harmonious labelings* [6], have also been studied. However, our research was the first to consider harmonious labelings for infinite graphs. In the next section, we will discuss which graphs we found to be harmonious and where our research into infinite graphs led us.

# 3 Harmonious

We attempted to find a way to build a finite harmonious graph in a way that could be continued indefinitely. This resulted in this type of graph (see Fig. 10) being harmonious.

**Definition 3.1.** $D_n$ is a graph on $2n+3$ vertices with $4n+3$ edges. $D_0$ is $(\{v_0, v_1, v_2\}, \{(v_0, v_1), (v_1, v_2), (v_0, v_2)\})$. For each $n > 0$ we add $v_{2n+1}$ and $v_{2n+2}$ and the edges $(v_{2n+1}, v_{2n+2})$, $(v_{2n}, v_{2n+1})$, $(v_{2n-1}, v_{2n+1})$, $(v_{2n}, v_{2n+2})$.

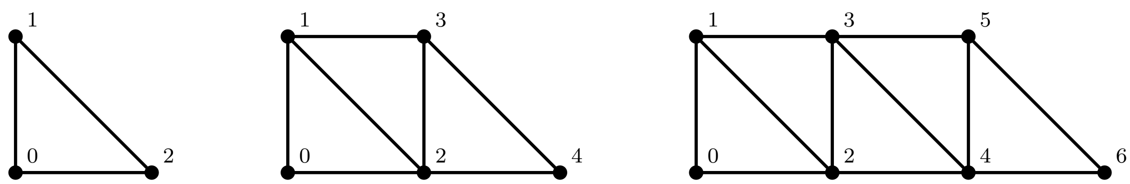


Figure 10: $D_0$, $D_1$, and $D_2$ from left to right.

**Theorem 3.1.** *All $D_n$ are harmonious.*

*Proof.* We will show that all $D_n$ are harmonious by showing that the graph can be built to an arbitrary length.

Stage 0: Add $v_0$; $\varphi(v_0) = 0$.

Stage 1: Add the vertices $v_1$ and $v_2$ along with the edges $(v_0, v_1)$, $(v_0, v_2)$, and $(v_1, v_2)$; $\varphi(v_1) = 1$ and $\varphi(v_2) = 2$.

Stage n $\geq$ 2: Add the vertices $v_{2n-1}$ and $v_{2n}$ along with the edges $(v_{2n-1}, v_{2n})$, $(v_{2n-2}, v_{2n})$, $(v_{2n-1}, v_{2n-2})$, and $(v_{2n-1}, v_{2n-3})$; $\varphi(v_{2n-1}) = 2n - 1$ and $\varphi(v_{2n}) = 2n$.

Note that the above procedure guarantees that at each stage the graph is harmonious. QED

The current definition of harmonious labeling being limited to finite graphs, we began to wonder how it could apply to infinite graphs and which infinite graphs would be harmonious.

**Definition 3.2.** An infinite graph $G = (V, E)$ has a *harmonious labeling* if there exists a bijection (see Definition 2.2) $\varphi$ from the vertices of G to the natural numbers such that the induced edge label $\varphi(v_i) + \varphi(v_j)$ for each edge is bijective to $\mathbb{Z}^+$.

Note that since the method used in the proof of Theorem 3.1 can be repeated indefinitely, the following graph is also harmonious.

**Definition 3.3.** The infinite graph $D = \bigcup_{n\in\mathbb{N}} D_n$.

Once we had a definition of a harmonious labeling for infinite graphs, we began to answer our second question of which infinite graphs are harmonious, starting with the simplest infinite graph, the semi-infinite path (see Fig. 11).

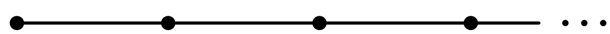

Figure 11: The semi-infinite path

**Theorem 3.2.** *The semi-infinite path is not harmonious.*

*Proof.* To show that the semi-infinite path is not harmonious, we will show that it is impossible for the numbers 1, 2, and 3 to all be induced edge labels on the path. Consider the edge label 1. The only way for an edge to be labeled 1 is for two adjacent vertices to be labeled 0 and 1. Next, consider the edge label 2. The only way for an edge to be labeled 2 is for two adjacent vertices to be labeled 0 and 2. Finally, the only way for an edge to be labeled 3 is either for two adjacent vertices to be labeled 0 and 3, or for two adjacent vertices to be labeled 1 and 2. Neither of these are possible, since 0 cannot be adjacent to three vertices and there can be no cycles in the semi-infinite path. Therefore, it is impossible to harmoniously label the semi-infinite path. QED

In the next section, we will define a new type of labeling that we came across in our research and show which graphs we found to have this new type of labeling.

# 4 Locally Harmonious

An interesting variation of the definition of harmonious labeling we found was the *locally harmonious labeling.*

**Definition 4.1.** A graph $G = (V, E)$ with $e$ edges has a *locally harmonious labeling* if there exists an injective function $\varphi$ from the vertices of G to the group of positive integers $\mathbb{Z}^+$ mod $e$ such that the induced edge label $\varphi(v_i) + \varphi(v_j)$ is in bijection to the set $\{0, \ldots, d(v_i) - 1\}$ mod $d(v_i)$ for each vertex $v_i$ (see Fig. 12).

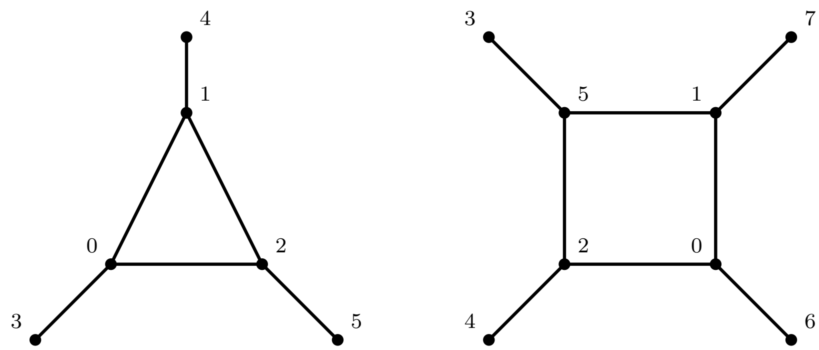


Figure 12: Examples of locally harmonious graphs

Again, we started our investigation of this type of labeling with the simplest finite graph, the path.


Figure 13: Locally Harmonious Labeling of $P_4$

**Lemma 4.1.** *All paths $P_{4n}$ are locally harmonious.*

*Proof.* We will show that all paths $P_{4n}$ are locally harmonious by induction.

Base case: $P_4$ is locally harmonious (see Fig. 13).

Now we will assume that $P_{4n}$ is harmonious.

Induction step: Now we will show $P_{4n+4}$ is locally harmonious. To show this, we must show that all labels $\{0, \ldots, 4n+3\}$ have been used and that the induced edge labels satisfy Definition 4.1. In the graph of $P_{4n}$, the vertex labels $\{0, \ldots, 4n-1\}$ must have been used. The vertices $v_{4n}$ and $v_{4n+1}$, as shown in Fig. 14, can be labeled as the two numbers in $\{4n, \ldots, 4n+3\}$ with opposite parity to $v_{4n-1}$. $v_{4n+2}$ and $v_{4n+3}$ can be labeled as the two with the same parity. Thus, all labels in $\{0, \ldots, 4n+3\}$ have been used, and the graph is locally harmonious. By induction, all $P_{4n}$ are harmonious. QED

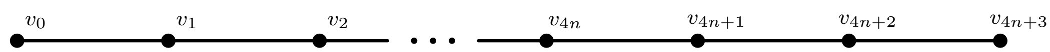


Figure 14: Locally Harmonious Labeling of $P_{4n+4}$

**Theorem 4.1.** *All paths are locally harmonious.*

*Proof.* Consider the path $P_n$.

When $n \equiv 1 \pmod 4$:

By Lemma 4.1, the path $P_{n-1}$ is locally harmonious. One end of $P_{n-1}$ will be labeled an even number, and the other end will be labeled an odd number, as can be seen from the proof of Lemma 4.1. To get $P_n$ from $P_{n-1}$, we add $v_n$ and the edge $(v_n, v_i)$, where $v_i$ is the vertex on the end labeled an odd number. $\varphi(v_n) = n - 1$. Now $v_i$ has an even and an odd labeled vertex adjacent to it, so it satisfies the definition of a locally harmonious labeling.

When $n \equiv 3 \pmod 4$:

Similarly, we know that the path $P_{n+1}$ is locally harmonious. Call the vertex in $P_{n+1}$ with label $n$ $v_i$, and the vertex on the end which is labeled an even number $v_j$. Switch the labels of these two vertices, and remove $v_j$, which is now labeled $n$. Since the labels we switch are both even, the graph is locally harmonious after the switch, as well as after the removal of $v_j$.

When $n \equiv 2 \pmod 4$:

Similar to the previous two cases, we know that the path $P_{n-2}$ is locally harmonious. Call the vertex on the end which is labeled an even number $v_i$ and the vertex on the end which is labeled an odd number $v_j$. Add the vertices $v_{n-1}$ and $v_n$, and the edges $(v_{n-1}, v_i)$ and $(v_n, v_j)$. $\varphi(v_{n-1}) = n-1$ and $\varphi(v_n) = n-2$. Now the vertices $v_i$ and $v_j$ have one even and one odd labeled vertex adjacent to them, so they still satisfy the definition of a locally harmonious labeling, as well as the new vertices $v_{n-1}$ and $v_n$ of degree one. See Fig. 15 for examples of each case. QED

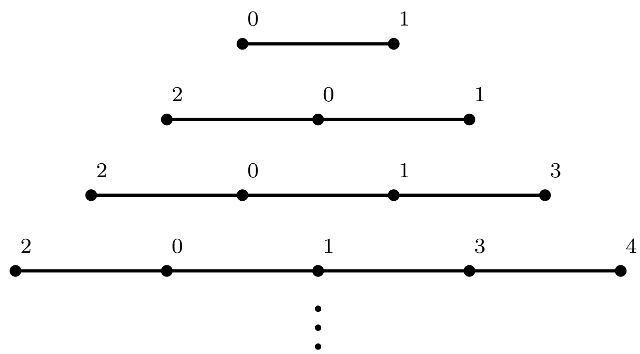


Figure 15: The paths $P_2$, $P_3$, $P_4$, and $P_5$ from top to bottom.

We also found that all complete bipartite graphs $K_{m,n}$ are locally harmonious.

**Theorem 4.2.** *All complete bipartite graphs are locally harmonious.*

*Proof.* First, divide the bipartite graph $K_{m,n}$ into the two disjoint sets described in Definition 2.6: $A = \{v_0, v_1, \ldots, v_{m-2}, v_{m-1}\}$ and $B = \{v_m, v_{m+1}, \ldots, v_{m+n-2}, v_{m+n-1}\}$ (see Fig. 16). The vertices in $A$ can be labeled with the labels $\{0, 1, \ldots, m-2, m-1\}$ and the vertices in $B$ can be labeled with the labels $\{m, m+1, \ldots, m+n-2, m+n-1\}$. Any vertex $v_i \in A$ is adjacent to all vertices in $B$, the labels of which are all distinct mod $n$. Therefore, all induced edge labels $\varphi(v_i) + \varphi(v_j)$ for all the vertices $v_j \in B$ will be distinct mod $n$. The same is true for all vertices in $B$, therefore the graph is locally harmonious. QED

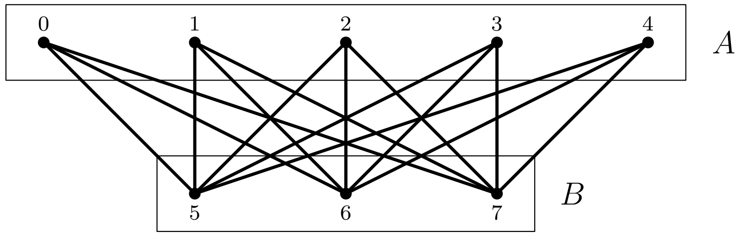


Figure 16: Locally Harmonious Labeling of $K_{5,3}$

As we did with harmonious labeling, we defined locally harmonious labeling for the case of infinite graphs.

**Definition 4.2.** An infinite graph $G = (V, E)$ has a *locally harmonious labeling* if there exists a bijection from the vertices of G to the natural numbers such that the induced edge label $\varphi(v_i) + \varphi(v_j)$ is in bijection to the set $\{0, \ldots, d(v_i) - 1\}$ mod $d(v_i)$ for each vertex $v_i$.

We found that for any finite locally harmonious graph with minimum degree one, there is an infinite graph which contains it as a proper subset that is also locally harmonious.

**Theorem 4.3.** *Assume $G$ is a finite locally harmonious graph with minimum degree one. There exists an infinite graph $\widetilde{G} \supset G$ which is locally harmonious.*

*Proof.* Consider a finite locally harmonious graph with minimum degree one G (see Fig. 17). Find a vertex of degree one; call it $v_i$ and its neighbor $v_j$. Now consider the semi-infinite path with vertices $\{p_0, p_1, \ldots\}$ and edges $\{(p_0, p_1), (p_1, p_2), \ldots\}$. Identify $v_i$ with $p_0$. Let $\varphi(p_1)$ be the least nonnegative integer not already used as a label such that it is of parity opposite to $\varphi(v_j)$. In general, let $\varphi(p_n)$ be the least nonnegative integer not already used as a label such that it is of parity opposite to $\varphi(p_{n-2})$. The resulting graph is an infinite locally harmonious graph which contains G (see Fig. 18). QED

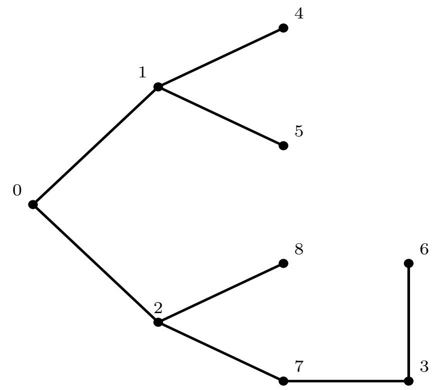


Figure 17: A finite locally harmonious graph $G$ with at least one vertex of degree one.

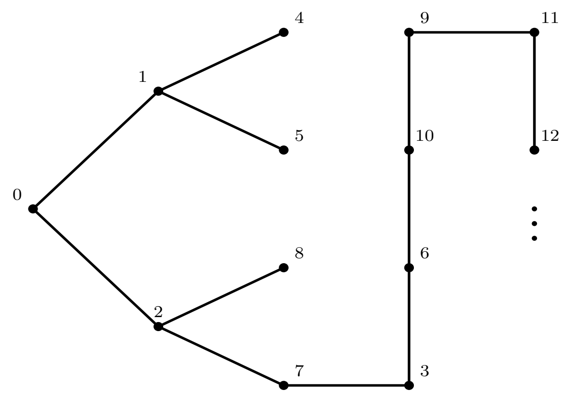


Figure 18: An infinite locally harmonious graph which contains $G$.

# 5 Conclusion

In Section 3, we were able to prove that all graphs $D_n$ are harmonious in a way that could be extended to prove that the infinite graph $D$ is harmonious. We also proved that the semi-infinite path is not harmonious. In Section 4, we were able to prove that all paths and complete bipartite graphs $K_{n,n}$ are locally harmonious. We also showed that for any finite locally harmonious graph G with minimum degree one, there is an infinite locally harmonious graph that contains G. This research helps to better understand infinite graphs and harmonious labeling, and opens up possibilities for new labelings such as the locally harmonious labeling.

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